7.3: Translation and Partial Fractions

Let us refresh ourselves on the rules of **partial fraction decomposition.** For the next two rules let

$$R(s) = \frac{P(s)}{Q(s)}$$

where $P(s)$ is a polynomial of degree less than that of $Q(s)$.

Rule 1. (Linear Factor Partial Fractions)
The portions of the partial fraction decomposition of $R(s)$ corresponding to the linear factor $s-a$ of multiplicity n is a sum of n partial fractions, having the form

$$\frac{A_1}{s-a} + \frac{A_2}{(s-a)^2} + \cdots + \frac{A_n}{(s-a)^n}$$

where $A_1, A_2, \ldots, A_n$ are constants.

Rule 2. (Quadratic Factor Partial Fractions)
The portions of the partial fraction decomposition of $R(s)$ corresponding to the quadratic factor $(s-a)^2+b^2$ of multiplicity n is a sum of n partial fractions, having the form

$$\frac{A_1 s + B_1}{(s-a)^2+b^2} + \frac{A_2 s + B_2}{((s-a)^2+b^2)^2} + \cdots + \frac{A_n s + B_n}{((s-a)^2+b^2)^n}$$

where $A_1, A_2, \ldots, A_n, B_1, B_2 \ldots, B_n$ are constants.

Theorem 1. (Translation of the s-Axis)
If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > c$, then $\mathcal{L}\{e^{at} f(t)\}$ exists for $s > a + c$, and

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a).$$

Equivalently,

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at} f(t).$$

Thus the translation $s \to s-a$ in the transform (s-axis) corresponds to multiplication of the original function of t by e^{at}.

Example 1. Find $\mathcal{L}\{e^{at} t^n\}$, $\mathcal{L}\{e^{at} \cos kt\}$ and $\mathcal{L}\{e^{at} \sin kt\}$.

Example 2. Consider a mass-and-spring system with $m = \frac{1}{2}$, $k = 17$ and $c = 3$ which gives the equation

$$x'' + 6x' + 34x = 0.$$

If the mass is set into motion with $x(0) = 3$ and $x'(0) = 1$, find the resulting damped free oscillations.

Example 3. Find the inverse Laplace transform of

$$R(s) = \frac{s^2 + 1}{s^3 - 2s^2 - 8s}.$$

Example 4. Solve the initial value problem

$$y'' + 4y' + 4y = t^2; \quad y(0) = y'(0) = 0.$$

Exercise 1. Solve the damped and forced mass-spring-dashpot system given by

$$x'' + 6x' + 34x = 30\sin 2t; \quad x(0) = x'(0) = 0.$$

Exercise 2. Solve the initial value problem

$$y^{(4)} + 2y'' + y = 4te^t; \quad y(0) = y'(0) = y''(0) = y^{(3)}(0) = 0.$$

Example 5. Solve the initial value problem

$$x'' + \omega_0^2 x = F_0 \sin \omega t; \quad x(0) = x'(0) = 0.$$

Homework. 1-21, 27-35 (odd)